(1) Fluid Mechanics II   Academic year: 2012/13

Advanced Fluid Mechanics Course

1 Gas Dynamics

1.1 Introduction:

The flow of compressible fluid is governed by the first law of thermodynamics, which relates to conservation of energy, and by the second law of thermodynamics, which relates heat interaction and irroversibity to entropy.
The flow is also affected by both kinematic and dynamic effects, which are discribed by Newton's laws of motion.

In all cases the flow mus fulfill the requirements of conservation of mass.

In general:

* Problems in which the direction, cross sectional area, and the shape of the duct do not change abruptly can be treated as one-dimensional, thereby providing a simple technique to generate sufficiently accurate solutions.

* External flows, such as a flow over a submerged object and interaction of a shock wave with the walls of a divergent passage, cannot be treated adequately using one-dimensional analysis.

1.2. The laws of thermodynamics.

The first law of thermodynamics deals with changes in energy distribution.

In a system, according to this law, heat interactions that occur during a complete cycle are related to work done.

$$\oint \delta Q + \oint dW = 0 \qquad (1)$$

Where δQ and δW represent infinitesimal amounts of heat and work and $\oint$ denotes a cyclic integral. the symbol δ indicates that the differential are in exact and the values of the differentials depend on the path followed between end States.

This is because both heat and work are not "point-functions" but "path-functions"

In a fixed system, when energy is exchanged between the system and its enviroment, the exchange in energy within the system is written as

$$\delta Q + \delta W = dE \qquad (2)$$

Where dE represents the Change in total energy of the system as a result of this process.

* Note: Energy is the Process.
Energy is a "Property"
Energy is a "Point-function"

In control volume the change of energy is due to heat and work interactions and also due to energy of the fluid as it crosses the boundary of the volume.

Enthalpy, H is defined to be a property representing the sum of internal energy U and flow work, PV.

Specific enthalpy

$$h = u + p.v \qquad (3)$$

where

v: represents the specific volume = $1/\rho$

p: pressure

u: specific internal energy [J/kg]

According to the second law of thermodynamics actual processes are irreversible.

* Note: A reversible process is a process in which a system and its environment can both be restored to their initial states. Such process can be performed only at an infinitely slow rate, so that system remains in quasi-equilibrium throughout the process.

A process which proceeds at a "finite" rate with finite potential differences is therefore irreversible.

Entropy, as we know, it is a measure of the probability of a system to be in a particular microscopic state and is associated with irreversibility of a ~~sys~~ ~~the~~ thermodynamic processes.

In a reversible process, the entropy change of a fixed system is due to heat interaction only.

$$dS = \left(\frac{\delta Q}{T}\right)_{rev.} \qquad (4)$$

On other hand, in an irreversible process the entropy change is due to heat interaction and irreversibilities, so that

$$dS > \left(\frac{\delta Q}{T}\right)_{irrev.} \qquad (5)$$

If the initial and final states are specified, real processes can occur only if the entropy change is larger than the value of $\frac{\delta Q}{T}$.
In a reversible adiabatic process the entropy remains constant and the process is called Isotropic.

By combining the first law with the second law, changes of entropy can be related to other state functions.

The following equations apply to process in one-component system in which gravity, motion, electricity, magnetic effects are absent

$$T\,ds = dU + p\,dV \qquad \text{constant } P \qquad (6)$$

$$T\,ds = dH - V\,dP \qquad \text{Constant } V \qquad (7)$$

When a substance obeys the perfect gas law there is a simple relation between C_p, C_v and R.

From the definitions of C_p and h (specific enthalpy), h only depends on T

$$C_p = \left(\frac{\partial h}{\partial T}\right)_p = \frac{dh}{dT} = \frac{d}{dT}\left(U + pv\right)$$

$$C_p = \frac{dU}{dT} + \frac{d}{dT}\left(RT\right) = \frac{dU}{dT} + R$$

$$C_p - \left(\frac{dU}{dT}\right) = R$$ and $\frac{dU}{dT}$ is equal to C_v at constant V

$$\boxed{C_p - C_v = R} \qquad (8)$$

We define another relation of specific heats, this is the ratio of specific heats γ, plays an important role in isentropic process.

$$\boxed{\gamma = \frac{C_p}{C_v}} \qquad (9)$$

from $C_p - C_v = R$ and $\gamma = \frac{C_p}{C_v}$

We obtain

$$C_p = \frac{\gamma R}{\gamma - 1} \quad \text{and} \quad C_v = \frac{R}{\gamma - 1}$$

From equation (6) we obtain the relation of s, u, v

$$ds = \frac{du}{T} + p\frac{dv}{T} \Rightarrow ds = C_v \frac{dT}{T} + R\frac{dv}{v}$$

and upon integrating

$$s_2 - s_1 = C_v . \ln\left(\frac{T_2}{T_1}\right) + R . \ln\left(\frac{v_2}{v_1}\right) \quad (11)$$

$$s_2 - s_1 = C_v \ln\left(\frac{T_2}{T_1}\right) * \left(\frac{v_2}{v_1}\right)^{\gamma-1} \quad (12)$$

Another alternative is that we can eliminate T or v from (12) with $pv = RT$, so we obtain.

$$s_2 - s_1 = C_v \ln\left(\frac{P_2}{P_1}\right) + C_p \ln\left(\frac{v_2}{v_1}\right)$$

$$s_2 - s_1 = C_v \ln\left(\frac{P_2}{P_1}\right) * \left(\frac{v_2}{v_1}\right)^{\gamma} \quad (13)$$

Also

$$s_2 - s_1 = C_p \ln\left(\frac{T_2}{T_1}\right) - R \ln\left(\frac{P_2}{P_1}\right) = C_v \ln\left(\frac{T_2}{T_1}\right)^{\gamma} * \left(\frac{P_2}{P_1}\right)^{-(\gamma-1)} \quad (14)$$

Often the isentropic process is taken as a model or as a limit for real adiabatic processes. If the entropy is constant at each step of the process, it follows that T and ρ, p and ρ and T and p are ~~constant~~ connected with each other during the process by the following laws.

$$\frac{T}{\rho^{\gamma-1}} = \text{const.} \qquad \frac{P}{\rho^{\gamma}} = \text{Const.} \qquad \frac{T^{\frac{\gamma}{\gamma-1}}}{P} = \text{const.}$$

For Isentropic processes the enthalpy change is important. It is calculated in terms of the initial temperature and the pressure ratio as

$$(\Delta h)_s = C_p (T_2 - T_1)_s$$

$$(\Delta h)_s = C_p . T_1 \left(\frac{T_2}{T_1} - 1 \right)_s$$

$$(\Delta h)_s = C_p . T_1 \left[\left(\frac{P_2}{P_1} \right)^{\frac{\gamma-1}{\gamma}} - 1 \right]$$

2. Concepts to Compressible Flow

2.1. Speed of Sound.

Pressure disturbances in the form of propagating waves are transmitted through a fluid as sucessive compressive and rarefaction waves because of the elastic nature of the fluid.

Sound waves are transmitted through a medium in a similar manner and the speed at which a small pressure pulse propagates through a compressible medium the speed of sound in that medium.

The speed of sound depends on the compressibility of the medium in which they propagate.

Consider a homogeneous compressible fluid of pressure P, density ρ and enthalpy h at rest in a one-dimensional passage of uniform cross-sectional area.

If the fluid is suddenly compressed from the left and infinitesimal pressure pulse will be generated and propagate to the right at a velocity **c.**

The fluid through which the wave front has passed is at a pressure $P + dP$, has a density $\rho + d\rho$ and moves to the right with a velocity dV.

The fluid on the right, into which the wave front is moving, has pressure P and a density ρ and is motionless

If we use a moving coordinate it looks as if a flow is entering from right at velocity c into the control volume and it is flowing out to the left a the velocity $c - dV$.

Applying continuity equation

$$\rho c A = (\rho + d\rho)(c - dV) A \quad (15)$$

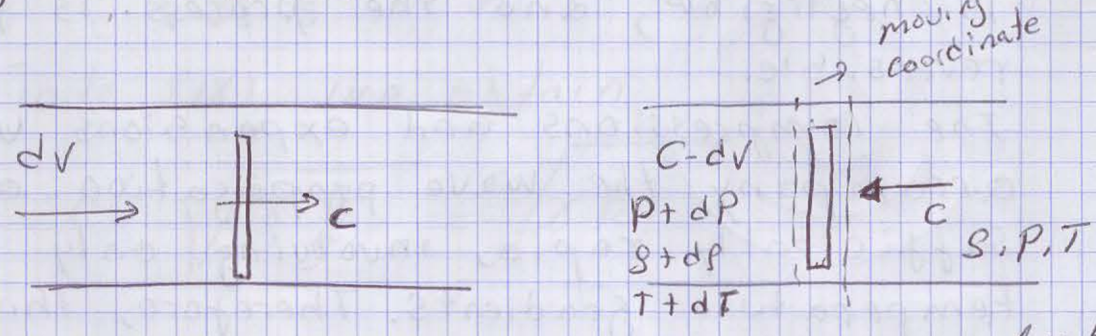


Figure 1 Disturbance moving from left

$$\rho c = \rho . c - \rho dV + d\rho . c - d\rho . dV$$

$$\rho dV = d\rho c$$

$$\left[\frac{d\rho}{\rho} = \frac{dV}{c} \right] \quad (15)$$

(9). The momentum equation is

$$PA - (P + dP)A = \rho AC(C - dV) - \rho AC.C$$

$$PA - PA - Adp = \rho AC^2 - \rho ACdV - \rho A.C^2$$

$$-Adp = \rho ACdV$$

$$\left[dP = \rho C dV\right] \quad (16)$$

Combining (15) and (16) we obtain

$$\frac{d\rho}{\rho} = \frac{dP}{\rho C.C} \Rightarrow \left[C^2 = \frac{dP}{d\rho}\right] \quad (17)$$

In this analysis, it was assumed that pressure wave is very weak, thus the amplitude of the pressure pulse is small, resulting in infinitesimal changes in fluid properties across the wave. Hence, the departure of the fluid from thermodynamics equilibrium is negligible, and the process is practically reversible.
The compressions and expansions which accompany the wave propagation are sufficiently rapid, involving only small temperature gradients. Therefore, this process is assumed to be adiabatic, so the process is both adiabatic and reversible, that is, isentropic.

Equation (17) is now given by

$$C = \sqrt{\left(\frac{\partial P}{\partial \rho}\right)_s} \quad (18)$$

(10) For perfect gas the relation between pressure and density in an isentropic process is given by

$$\frac{P}{\rho^{\gamma}} = \text{constant}$$

Taking logarithm, differentiating.

$$\ln\left(\frac{P}{\rho^{\gamma}}\right) = \ln(\text{constant})$$

$\ln p - \gamma \ln \rho = \text{Constant}$ then differentia.

$$\frac{dP}{P} - \gamma \frac{d\rho}{\rho} = 0$$

$$\left[\frac{dP}{P} = \gamma \frac{d\rho}{\rho} \right] \quad (19)$$

From the equation of state.

$$\left(\frac{\partial P}{\partial \rho}\right)_s = \gamma \frac{P}{\rho} = \gamma R T \quad (20)$$

(20) into (18) we obtain

$$C = \sqrt{\gamma \frac{P}{\rho}} = \sqrt{\gamma R T} \quad (21)$$

Ex.

Calculate the velocity of sound at air at 300K and 650 K $\gamma = 1.4$ at 300 K

$$C = \sqrt{(1.4) \cdot \frac{8314}{28,97} \times \frac{N.m}{kg.K} \cdot (300)K} = 347 \, m/s$$

at 650K $\gamma = 1,397$

$C = 506 \, m/s.$

2.2 Mach number

When a body moves through a fluid or when a fluid flows past a body or within the walls of a duct, each element of solid surface tends to divert the fluid from the course.

Let us consider a point source of disturbance moving at a uniform linear speed through a compressible medium.

At each instant of time the point source may be imagined to emit an infénitesimal pressure wave which spreads spherically from the point of emission with speed of sound relative to the fluid.

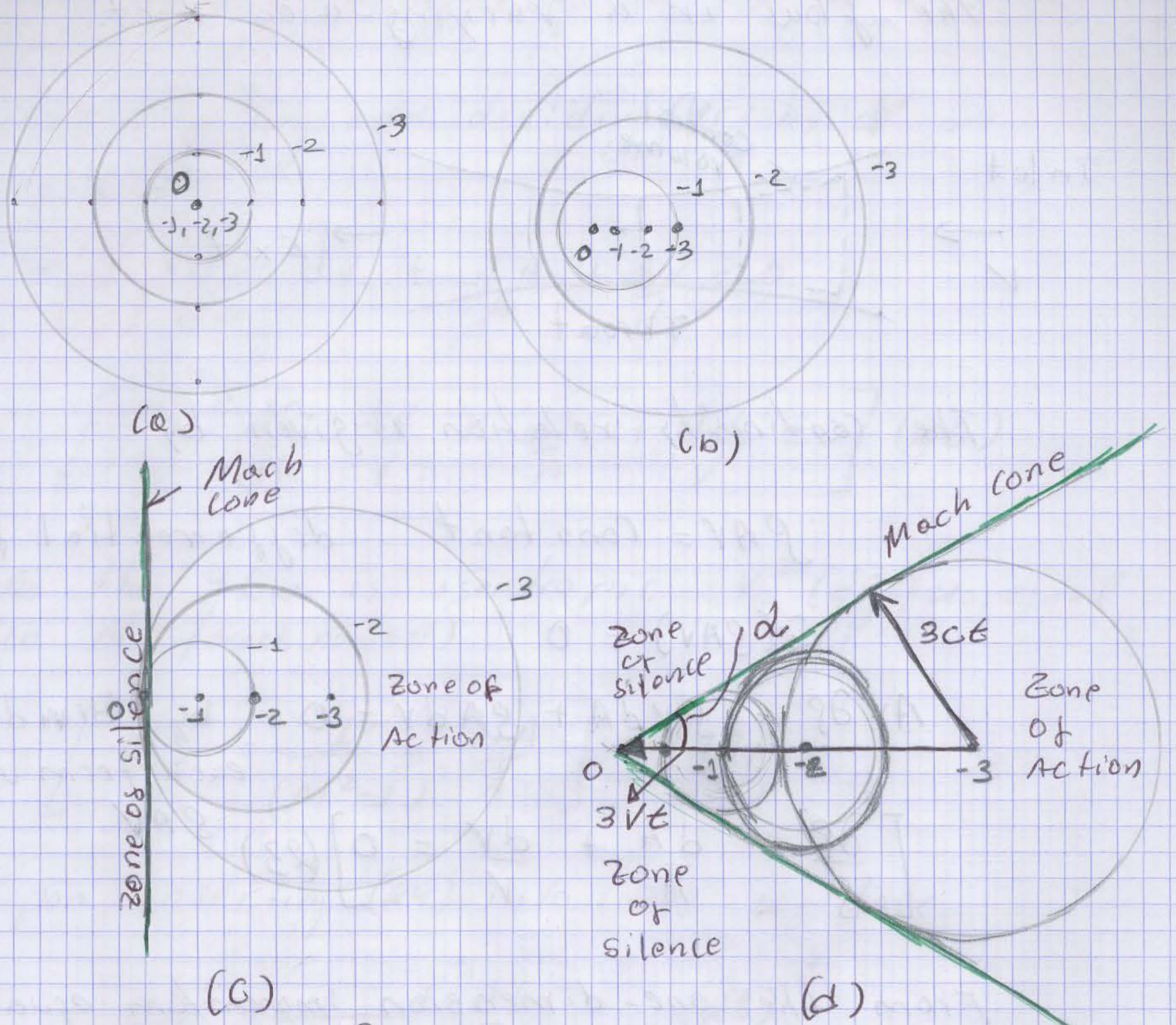
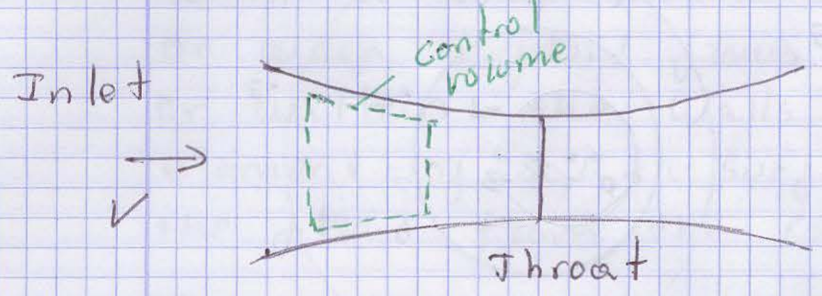


Figure 2. Pressure field of moving source

Mach number is defined to be the dimensionless parameter which indicates the relative velocity of fluid to speed of sound

$$M = \frac{V}{c} = \frac{V}{\sqrt{\left(\frac{\partial p}{\partial \rho}\right)_s}} \qquad (22)$$

(13). The figure shows the control volume of the flow in a varying-area duct.



The continuity relation is given by

$$\rho A V = \text{Constant} \quad \text{differential form}$$

$$\partial(\rho A V) = 0$$

$$A V \partial\rho + \rho V dA + \rho A dV = 0$$ by dividing each term with $\rho A V$

$$\left[\frac{\partial\rho}{\rho} + \frac{dA}{A} + \frac{dV}{V} = 0\right] (23)$$

From the one-dimension momentum equation

$$V \frac{dV}{dx} = -\frac{1}{\rho}\frac{dP}{dx}$$

$$V \frac{dV}{dP} = \frac{1}{\rho} \quad (24)$$

Dividing (24) by $d\rho/d\rho$, and from (23) and (24) we obtain

$$V \frac{dV}{dp} d\rho = -\frac{d\rho}{\rho}$$

$$V \frac{dV}{dp} \cdot d\rho = \frac{dA}{A} + \frac{dV}{V}$$

$$-V.\frac{d\rho}{dp}.dV + \frac{dA}{A} + \frac{dV}{V} = 0 \quad (25)$$

multiplying and dividing (25) by V we obtain

$$-\left(\frac{V.}{V}\right).V\frac{d\rho}{dp}.dV + \frac{dA}{A} + \frac{dV}{V} = 0$$

$$\left[-V^2\frac{d\rho}{dp}.\frac{dV}{V} + \frac{dA}{A} + \frac{dV}{V} = 0\right] \quad (26)$$

As the flow is isentropic $V^2\left(\frac{d\rho}{dp}\right)$ is equal to M^2 (Mach number)

$$\left[\frac{dV}{V} = \frac{1}{(M^2-1)}\frac{dA}{A}\right] \quad (27)$$

Upon inserting (24) into (27) we have

$$\frac{dp}{\rho V^2} = \frac{1}{(1-M^2)}.\frac{dA}{A} \quad (28)$$

Also density is related to area or velocity in the following way.

$$\frac{d\rho}{\rho} = \frac{M^2}{(1-M^2)}.\frac{dA}{A} = M^2\frac{dV}{V} \quad (29)$$

Finally

$$\left[\frac{dV/V}{dA/A} = \frac{1}{M^2-1}\right] \quad (30)$$

This relation is quite important to understand the pattern of flow with respect to the Mach number.

If $M < 1$, the ratio is negative and the velocity varies inversely with cross-sectional area.

In other words the flow accelerates through a converging duct while it decelerates Through a diverging duct.

If $M > 1$ The ratio is positive and the velocity varies in the same sense as the cross-sectional area.

Supersonic flow, flow accelerates in a diverging duct and decelerates in converging duct.

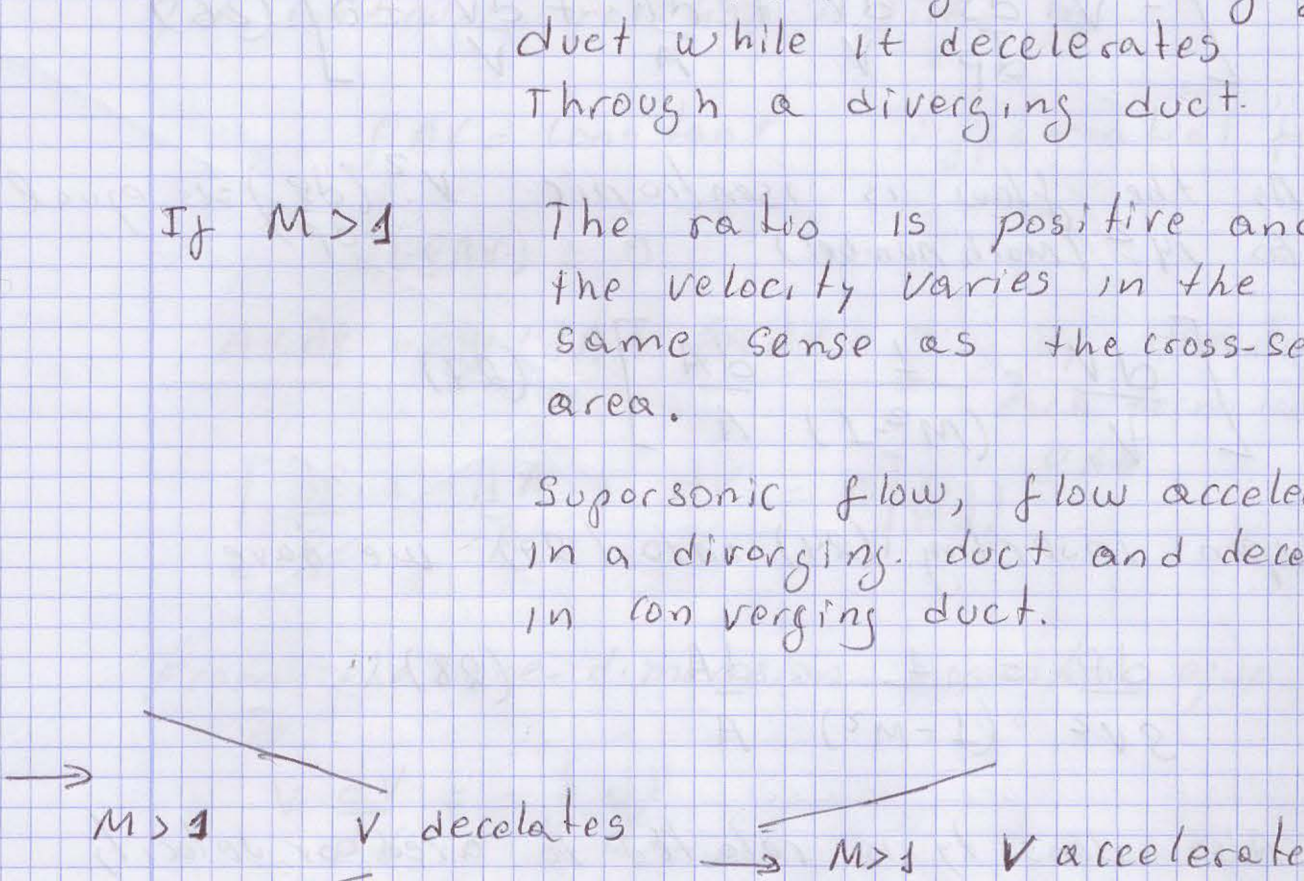


Figure 3 Effects of area change in supersonic and subsonic flows.

From the relation of enthalpy and temperature $h = C_p T$ and stagnation point $h_o = h + \frac{V^2}{2}$

$$\left[\frac{T_o}{T} = 1 + \frac{V^2}{2 C_p T}\right] \quad (31)$$

or

$$\left[\frac{T_o}{T} = 1 + \frac{\gamma - 1}{2} M^2\right] \quad (32)$$

where we used the relations

$$\frac{V^2}{\gamma R T} = M^2 \quad \text{and} \quad C_p = \frac{\gamma R}{\gamma - 1}$$

Equation (32) is valid for adiabatic flow. Note that T_o is the same for all points in the flow, provided the flow is adiabatic

For Isentropic flow.

$$\frac{P_o}{P} = \left(\frac{T_o}{T}\right)^{\frac{\gamma}{\gamma-1}} \quad (33)$$

And

$$\frac{\rho}{\rho_o} = \left(\frac{T_o}{T}\right)^{\frac{1}{\gamma-1}} \quad (34)$$

Combining these equations with (32) we obtain.

$$\frac{P_o}{P} = \left(1 + \frac{(\gamma - 1)}{2} M^2\right)^{\frac{\gamma}{\gamma-1}} \quad (35)$$

And $$\frac{\rho}{\rho_o} = \left(1 + \frac{(\gamma - 1)}{2} . M^2\right)^{\frac{1}{\gamma-1}} \quad (36)$$

The speed of sound is also a function of temperature.

$$\frac{C_0}{C} = \left(\frac{T_0}{T}\right)^{1/2} = \left(1 + \left(\frac{\gamma-1}{2}\right)M^2\right)^{1/2} \quad (37)$$

Where C_0 is the speed of sound when the gas is at the stagnation temperature

The value of fluid properties when the flow is at $M = 1$ are called Critical values and are denoted by an asterisk * to distinguish them from properties to other Mach numbers.

The critical values referred to stagnation properties are give as.

$$\frac{T^*}{T_0} = \left(\frac{C^*}{C_0}\right)^2 = \frac{2}{\gamma+1} \quad (38)$$

$$\frac{P^*}{P_0} = \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma+1}} \quad (39)$$

$$\frac{\rho^*}{\rho_0} = \left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}} \quad (40)$$

$$\frac{C^*}{C_0} = \left(\frac{2}{\gamma+1}\right)^{1/2} \quad (41)$$

Properties of the fluid refered to critical properties are obtained as.

$$\frac{T}{T^*} = \frac{\gamma+1}{2\left(1 + \frac{\gamma-1}{2}M^2\right)} \quad (42)$$

$$\frac{P}{P^*} = \left[\frac{\gamma+1}{2\left(1 + \frac{\gamma-1}{2}M^2\right)}\right]^{\frac{\gamma}{\gamma-1}} \quad (43)$$

$$\frac{\rho}{\rho^*} = \left[\frac{\gamma+1}{2\left(1+\frac{\gamma-1}{2}M^2\right)}\right]^{\frac{1}{\gamma-1}} \quad (44)$$

$$\frac{c}{c^*} = \left[\frac{\gamma+1}{2\left(1+\frac{\gamma-1}{2}M^2\right)}\right]^{1/2} \quad (45)$$

+ Consider the flow of a perfect gas through a converging nozzle.

+ The nozzle discharges into large chambers, which the pressure P_b can be regulated.

+ Let P_e be the exit pressure just inside the nozzle. and subscript o denote stagnation conditions

+ If P_b is below P_o the flow occurs.

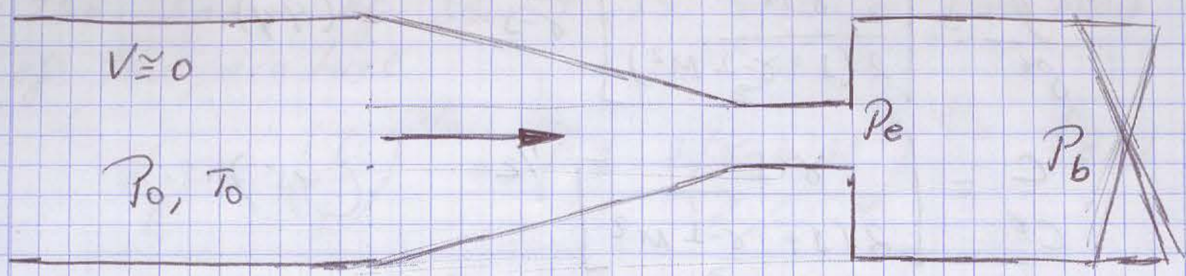
+ The amount of mass flow rate increases as P_b decreases. But the increase of mass flow rate stops at a certain ratio of pressures.

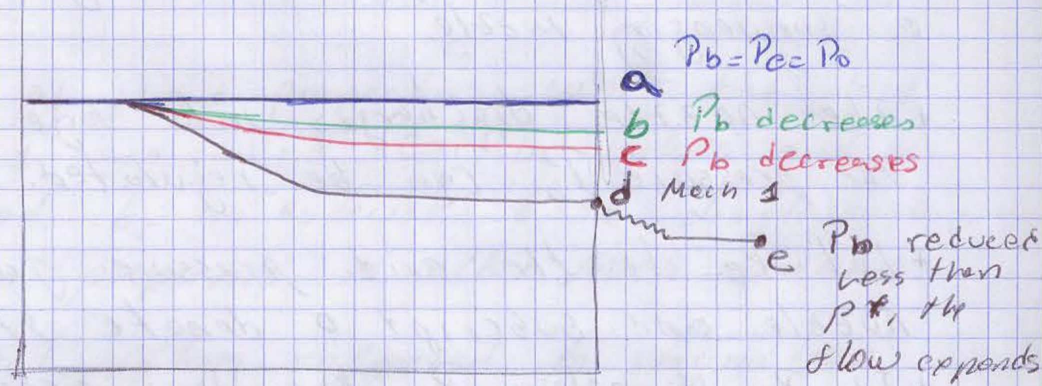
+ The pressure, which corresponds to the maximum rate of flow, is the critical P^*_c.

+ For exit pressure greater than or equal to the critical pressure, the back pressure is equal to the exit pressure. and flow in the nozzle is able to sense changes in back pressure.

+ If the back pressure is lower than exit pressure, the critical pressure and the conditions upstream of the nozzle exit as well as the mass flow rate are not affected. this corresponds a nozzle is choked. and the changes of back pressure are not sensed upstream of the nozzle exit.

+ The value of P_e/P_o, where P_o is the upstream pressure, cannot be made less than the critical pressure ratio unless there is a throat upstream of the exit section.

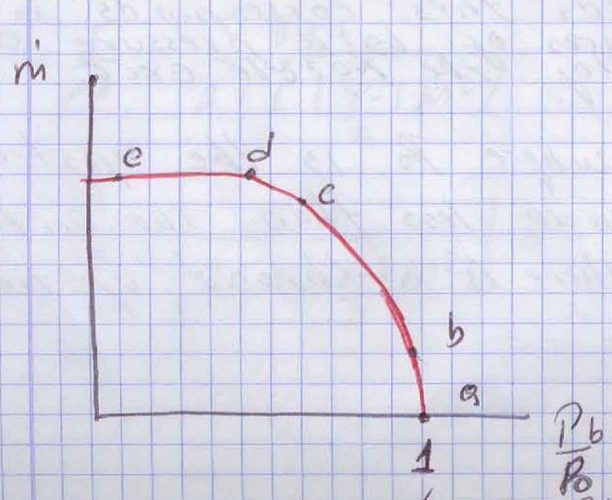




(a)

Figure 4. Effect of back pressure on the operation of a converging nozzle.

From the figure Case (a) corresponds to $P_b = P_e = P_o$ case (b) and (c) corresponds to a decreasing P_b the case (d) corresponds a decreasing the P_b in which results in greater mass flow rates and new pressure variation within nozzle, then the Mach number increases as P_b decreases, however and eventually a Mach number of unity will be attained at the nozzle exit.
Case (e) corresponds to a P_b less than P^* and the nozzle is choked.



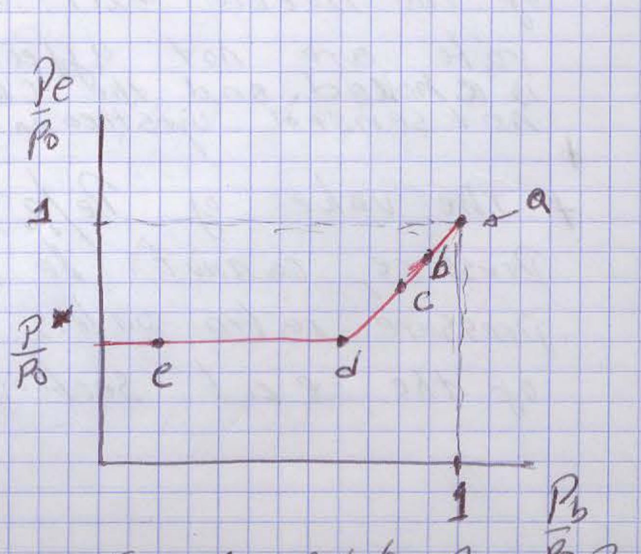
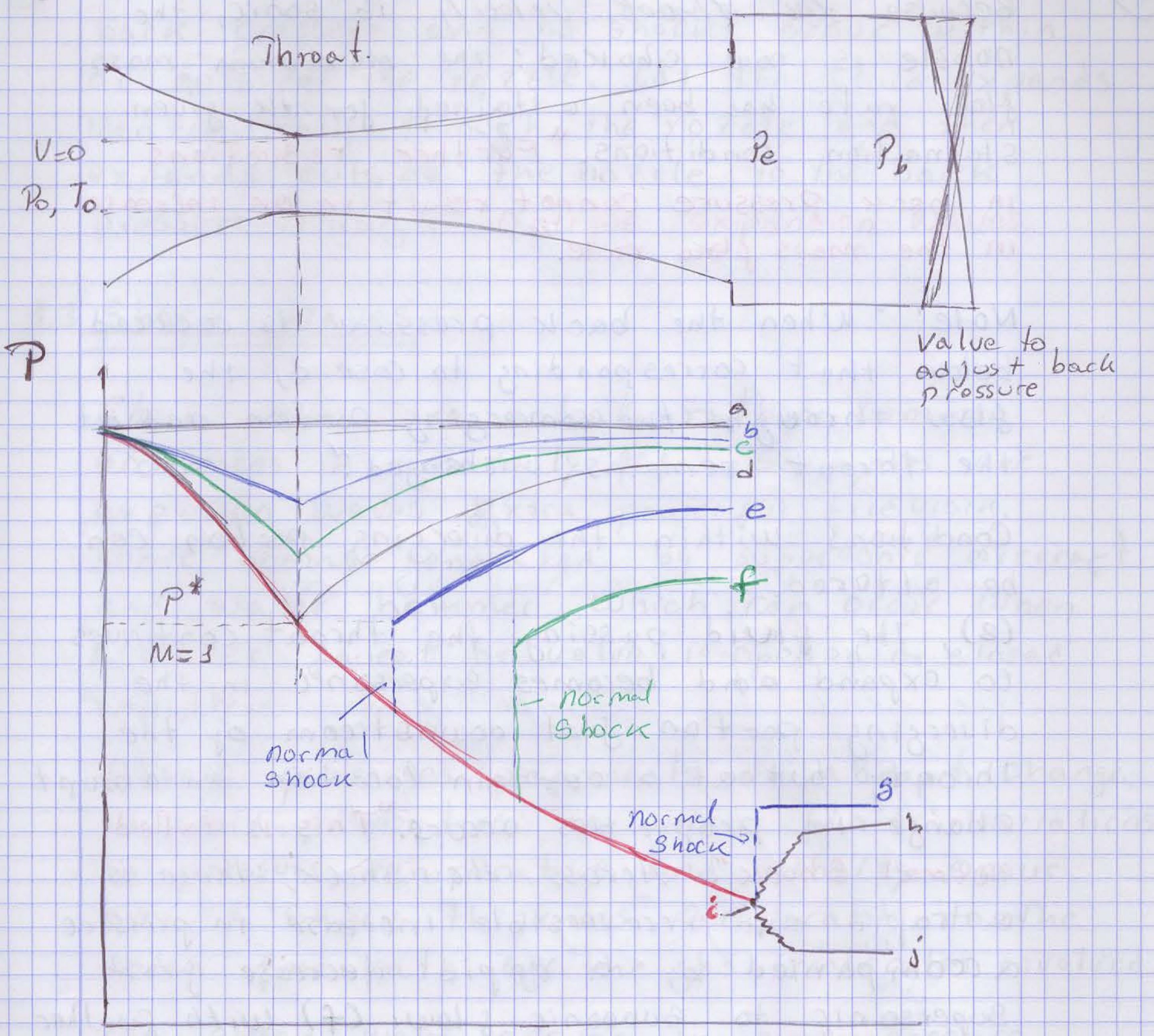


Figure 5. Mass flow rate vs $P_b/$ ... Figure 6. Relation Pe ... Po Pa



Case a, b, c, and d, (a) Correspond to $P_b = P_e = P_o$ which there is no flow. (b) When the back pressure is slightly less than P_o, there is some flow and the flow is subsonic throughout the nozzle. At the throat velocity increases and pressure decreases. And the diverging acts as diffuser in which pressure increases and velocity decreases. (c) when back pressure is reduced further, the mass flow rate and velocity at the throat are greater than before, but the flow remains subsonic. (d) As the back pressure is reduced, the Mach number at the throat increases, and eventually a Mach number of unity is attained at throat.

(21)

Because the throat velocity is sonic, the nozzle is now choked: the maximum mass flow rate has been attained for the given stagnation conditions. "Further reductions in back pressure cannot result in an increase in the mass flow rate."

Note: "When the back pressure is reduced below that corresponding to case d, the flow through the converging portion and at the throat remains unchanged."

Conditions within the diverging portion can be altered.

(e), The fluid passing the throat continues to expand and becomes supersonic in the diverging portion just downstream of the throat; but at a certain location an abrupt change in properties occurs. "This is called normal shock". Across the shock, there is a rapid and irreversible increase in pressure accompanied by a rapid decrease from supersonic to subsonic flow. (f) with further reductions in back pressure, the shock location moves farther downstream of the throat until it stands at the exit. (g). Shock location at the exit.

In the case h, i, and j where the back pressure is less than that corresponding to case g. In each of these cases, the flow through nozzle is not affect. (h) pressure decreases continuously as the fluid expands isentropically through the nozzle. and increases the back pressure outside nozzle. The compression that occurs outside the nozzle involves Oblique Shock Waves.

(22) (i), the fluid expands isentropically to the back pressure and no shocks occur within or outside the nozzle. (j) the fluid expands isentropically through the nozzle and then expands outside the nozzle to the back pressure through Oblique expansion waves.

2.3 Shock Waves

Shock waves are actually rather common occurrences in our daily life. Examples are explosion waves extra powerful fire work, sonic booms generated by supersonic aircraft and water hammer which can occur when a water faucet or valve is opened or closed rapidly.

Shock processes represents an abrupt change in fluid properties, in which finite variations in pressure, temperature and density occur over a shock thickness comparable to the mean free path of the gas molecules involved.

The shock wave occurs in a supersonic flow mainly because supersonic flow adjusts to the presence of a body by means of shock waves. This wave does not appear in subsonic flows because subsonic flows can adjust by gradual changes in flow properties.

In a normal or one-dimensional shock the change of properties occurs in the same direction as that of the flow. whereas in an oblique or a multidensional shock, the change of properties does not necessary occur in the flow direction.

Reviewing the second law of thermodynamics a compression shock involves an entropy increase resulting a loss in stagnation pressure.

It is sufficiently discussed by Shapiro that the formation of a compression shock is the result of accumulation of succession of sound waves. And also, it is discussed that an expansion or rarefraction shock never forms.

Consider a normal shock propagating adiabatically through a gas in a duct of constant cross-sectional area A.

Assuming that the gas is perfect and that the isentropic exponent γ does not vary with temperature.

Steady state flow conditions and continuity equation is applied we obtain

$$\frac{\dot{m}}{A} = \rho_1 V_1 = \rho_2 V_2 \quad (46)$$

The momentum equation no friction we obtain.

$$P_1 - P_2 = \frac{\dot{m}}{A}(V_2 - V_1) \quad (47)$$

Energy equation for adiabatic steady flow

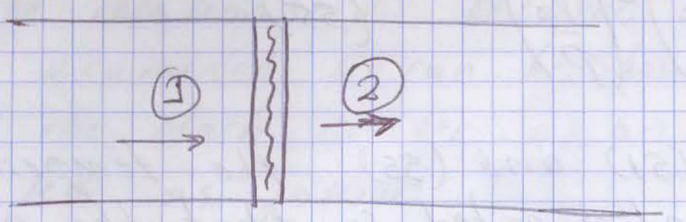
$$h_{01} = h_{02} = h_1 + \frac{V_1^2}{2} = h_2 + \frac{V_2^2}{2} \quad (48)$$

the flow accross the shock is adiabatic and irreversible,

$$S_2 - S_1 > 0 \quad (49)$$

Now the properties of the gas downstream of the shock are obtained first

$$T_{01} = T_{02}$$



$V_1 > V_2$

$T_1 < T_2$

$P_1 < P_2$

$S_1 < S_2$

Figure 6 Changes of properties across a normal shock.

The stagnation temperatures are

$$T_0 = T\left(1 + \frac{\gamma - 1}{2} M^2\right) \qquad (50)$$

So the ratio of static temperature is

$$\frac{T_2}{T_1} = \frac{1 + \frac{\gamma-1}{2} M_1^2}{1 + \frac{\gamma-1}{2} M_2^2} \qquad (51)$$

From the continuity equation and the equation of state.

$$\frac{T_2}{T_1} = \frac{\rho_1 P_2}{\rho_2 P_1} = \frac{V_2 P_2}{V_1 P_1} \qquad (52)$$

Velocity ratio is

$$\frac{V_2}{V_1} = \frac{M_2}{M_1}\sqrt{\frac{T_2}{T_1}} \qquad (53)$$

Replacing (53) into (52) we obtain.

$$\frac{T_2}{T_1} = \frac{M_2}{M_1}\sqrt{\frac{T_2}{T_1}} \cdot \frac{P_2}{P_1} \qquad (54)$$

And $$\frac{T_2}{T_1} = \left(\frac{M_2}{M_1}\right)^2 \cdot \left(\frac{P_2}{P_1}\right)^2 \qquad (55)$$

by combining (51) and (55) the temperature ratio can be eliminated so that the pressure ratio across the shock is

$$\frac{P_2}{P_1} = \frac{M_1}{M_2}\sqrt{\frac{1+\frac{\gamma-1}{2}M_1^2}{1+\frac{\gamma-1}{2}M_2^2}} \qquad (56)$$

The density ratio across the shock has been shown to be.

$$\frac{\rho_2}{\rho_1} = \frac{P_2 T_1}{P_1 T_2} \qquad (57)$$

Combining (53) and (57) the density ratio becomes

$$\frac{\rho_2}{\rho_1} = \frac{M_1}{M_2}\sqrt{\frac{1+\frac{\gamma-1}{2}M_2^2}{1+\frac{\gamma-1}{2}M_1^2}} \qquad (58)$$

The ratio of stagnation pressures is given by

$$\frac{P_{02}}{P_{01}} = \frac{\frac{P_{02}}{P_2}\cdot P_2}{\frac{P_{01}}{P_1}\cdot P_1} \qquad (59)$$

(annotations in red: $\frac{P_{02}}{P_2} \cdot P_2 = 1$, $\frac{P_{01}}{P_1} \cdot P_1 = 1$)

$$\frac{P_{02}}{P_{01}} = \frac{M_1}{M_2}\left(\frac{1+\frac{\gamma-1}{2}M_2^2}{1+\frac{\gamma-1}{2}M_1^2}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \qquad (60)$$

We need to analize the entropy change across the shock and we obtain this expression

$$S_2 - S_1 = C_p \cdot \ln\left(\frac{T_2}{T_1}\right) - R \cdot \ln\left(\frac{P_2}{P_1}\right) = C_p \ln\left(\frac{T_2/T_1}{(P_2/P_1)^{\frac{\gamma-1}{\gamma}}}\right) \qquad (61)$$

The relationship between M_1 and M_2 will be now examined. It can be shown that the velocity and the Mach number of an ideal gas are related as follows.

$$\rho V^2 = \gamma P M^2 \qquad (62)$$

Accordingly, the momentum equation is given as

$$P_1 + \gamma P_1 M_1^2 = P_2 + \gamma P_2 M_2^2 \qquad (63)$$

or

$$\frac{P_2}{P_1} = \frac{1 + \gamma M_1^2}{1 + \gamma M_2^2} \qquad (64)$$

Using the equation (55) and (64) we obtain

$$\frac{M_1 \sqrt{1 + \left(\frac{\gamma-1}{2}\right) M_1^2}}{1 + \gamma M_1^2} = M_2 \frac{\sqrt{1 + \frac{\gamma-1}{2} M_2^2}}{1 + \gamma M_2^2} \qquad (65)$$

In order to solve this relation it is convenient to define a function.

$$F(M) = M^2 \frac{\left(1 + \frac{\gamma-1}{2} M^2\right)}{\left(1 + \gamma M^2\right)^2} \qquad (66)$$

* It is noticed that $F(M)$ has a peak value at $M=1$ where $T_1 = T_2$ and $P_1 = P_2$.
* For each value of the function $F(M)$ there is a corresponding Mach number upstream of the shock and a corresponding Mach number downstream of the shock.

And we obtain the relation of the Mach numbers.

$$M_2^2 = \frac{\left(M_1^2 + \frac{2}{\gamma-1}\right)}{\left(\frac{2\gamma}{\gamma-1} M_1^2 - 1\right)} \qquad (67)$$

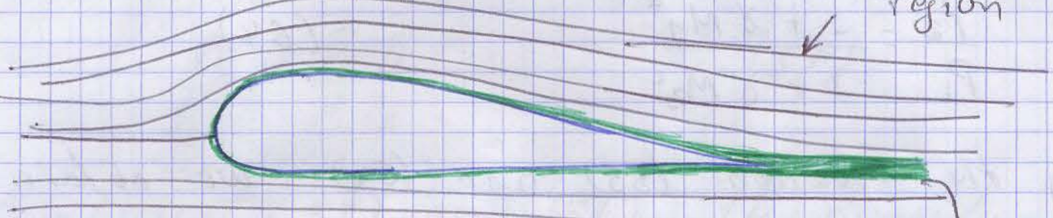
2.4 Method of characteristics

This analysis is limited to two dimensional, irrotational, isentropic, supersonic flow.

Recalling

Irrotational flow

There are ~~a~~ Regions of flow in which fluid particles have not net rotation; these regions are called Irrotational.



In general, inviscid regions of flow far away from solid walls and wakes of bodies are also irrotational.
Point out, there are situations in which an inviscid region of flow may not be irrotational (solid body rotation)

Solutions obtained for the class of flow defined by irrotationality are thus approximations of full Navier-Stokes solutions

Mathematically, the approximation is that vorticity is negligibly small.

Irrotational approximation.

$$\vec{\zeta} = \vec{\nabla} \times \vec{V} \cong 0 \qquad (68)$$

$$\text{Curl}(\vec{V}) = 0 \qquad (69)$$

(28) Continuity equation

Defining vector indetity $\vec{\nabla}\times\vec{\nabla}\phi = 0$ and $\vec{\nabla}\times\vec{V} = 0$ then $\vec{V} = \vec{\nabla}\phi$

Proving $\vec{\nabla}\times\vec{\nabla}\phi = 0$

$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix} = \left(\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y}\right)i + \left(\frac{\partial^2\phi}{\partial z\partial x} - \frac{\partial^2\phi}{\partial x\partial z}\right)j + \left(\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\phi}{\partial z\partial x}\right) = 0$$

If the curl of a vector is zero, the vector can be expressed by a gradient of a scalar function ϕ, called the potential function.

For irrotational regions of flow (also called regions of potential flow).

$$\vec{V} = \vec{\nabla}\phi \quad (70)$$

This not restricted to two-dimensional flow, this can be applied for three-dimensional flow.

$$U = \frac{\partial\phi}{\partial x} \quad V = \frac{\partial\phi}{\partial y} \quad W = \frac{\partial\phi}{\partial z}$$

applying ∇ in (70)

$$\vec{\nabla}\cdot\vec{V} = \vec{\nabla}\cdot\vec{\nabla}\phi = 0$$

$$\vec{\nabla}^2\phi = 0 \quad (71)$$

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0$$

this is called Laplace equation and is valid only in regions where the irrotational flow approximation is reasonable

(29) Momentum equation for irrotational flow.

We have defined Navier-Stoke equation to solve pressure field.

$$\rho\left[\frac{\partial \vec{V}}{\partial t} + (\vec{V}\cdot\vec{\nabla})\vec{V}\right] = -\vec{\nabla}P + \rho\vec{g} + \mu\nabla^2\vec{V} \quad (72)$$

The viscous term of equation can be analyzed

$$\mu\nabla^2(\vec{V}) = \mu\nabla^2(\nabla\phi) = \mu\cdot\nabla(\underbrace{\nabla^2\phi}_{0}) = 0$$

irrotational region flow.

The Navier-Stoke equation is reduced to Euler equation.

$$\rho\left(\frac{\partial \vec{V}}{\partial t} + (\vec{V}\cdot\vec{\nabla})\vec{V}\right) = -\vec{\nabla}P + \rho\vec{g} \quad (73)$$

$\frac{\partial \vec{V}}{\partial t} = 0$ steady incompressible; $(\vec{V}\cdot\vec{\nabla})\vec{V} = \vec{\nabla}\left(\frac{V^2}{2}\right) - \vec{V}\times\vec{\zeta}$; $\vec{g} = \vec{\nabla}(-gz)$ acting only in z-directions.

$$\vec{\nabla}\left(\frac{P}{\rho} + \frac{V^2}{2} + gz\right) = 0 \quad (74)$$

We now can say that if a gradient of a scalar quantity is zero everywhere, the scalar quantity itself must be a constant.

$$\frac{P}{\rho} + \frac{V^2}{2} + gz = C = \text{constant everywhere} \quad (75)$$

steady incompressible Bernoulli equation in irrotational regions of flow.

Stream function

Consider the case of incompressible, two-dimensional flow

Continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (76)$$

this equation can be transformed in one dependent variable (ψ) instead of two dependent variable (u and v), wher (ψ) is stream function.

$$\left.\begin{array}{l} u = \frac{\partial \psi}{\partial y} \text{ and} \\ v = -\frac{\partial \psi}{\partial x} \end{array}\right\} (77)$$

Replacing (77) into (76) we obtain.

$$\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{-\partial \psi}{\partial x}\right) = 0$$

$$\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0$$

Continuing with the method of characteristics assuming that the fluid is a perfect gas and the flow is steady with negligible gravity forces.

The continuity equation for two-dimensional steady flow can be given in terms of the velocity potential as.

$$\frac{\partial}{\partial x}\left(\rho \frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\rho . \frac{\partial \phi}{\partial x}\right) = 0 \quad (78)$$

The momentum equation in term of ϕ

$$\rho\left(\frac{\partial \vec{V}}{\partial t} + (\vec{V} . \vec{\nabla}) . \vec{V}\right) = -\vec{\nabla} P + \cdots$$

$\frac{\partial \vec{V}}{\partial t} = 0$ steady state

irrotational and gravity zero.

$\mu \nabla^2 V = 0$

we obtain.

$$dp = -\rho\, d\left(\frac{V^2}{2}\right) \quad (79)$$

$$dp = -\rho\, d\left(\frac{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2}{2}\right) \quad (80)$$

we know that speed of sound as given by

$$c^2 = \left(\frac{dp}{d\rho}\right)_s \quad (81)$$

$d\rho = \frac{dp}{c^2}$ => from (80) we obtain

$$d\rho = -\frac{\rho}{c^2}\, d\left(\frac{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2}{2}\right) \quad (81)$$

$$\frac{\partial\rho}{\partial x} = \frac{-\rho}{c^2}\left(\frac{\partial\phi}{\partial x}\cdot\frac{\partial^2\phi}{\partial x^2} + \frac{\partial\phi}{\partial y}\cdot\frac{\partial^2\phi}{\partial x\partial y}\right) \quad (82)$$

$$\frac{\partial\rho}{\partial y} = \frac{-\rho}{c^2}\left(\frac{\partial\phi}{\partial x}\frac{\partial^2\phi}{\partial x\partial y} + \frac{\partial\phi}{\partial y}\cdot\frac{\partial^2\phi}{\partial y^2}\right)$$

Substituting (82) into 78 we obtain.

$$(83) \quad \left[c^2 - \left(\frac{\partial\phi}{\partial x}\right)^2\right]\frac{\partial^2\phi}{\partial x^2} + \left[c^2 - \left(\frac{\partial\phi}{\partial y}\right)^2\right]\frac{\partial^2\phi}{\partial y^2} - 2\frac{\partial\phi}{\partial x}\cdot\frac{\partial\phi}{\partial y}\cdot\frac{\partial^2\phi}{\partial x\partial y} = 0$$

and we know that

$$c_p T = \frac{c_p}{\gamma R}\gamma R T = \frac{c_p}{\gamma R}\cdot c^2 = \frac{1}{\gamma - 1}c^2 \quad (84)$$

(32) we have defined that relation $\frac{C_0}{C_0}$ on C is given by.

$$\frac{C_0}{C} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{1/2} \quad (85)$$

$$\frac{C_0^2}{C^2} = 1 + \frac{\gamma - 1}{2} \frac{V^2}{C^2}$$ finaly we obtain

$$V^2 + \frac{2}{\gamma - 1} c^2 = \frac{2}{\gamma - 1} C_0^2$$ and.

$$c^2 = C_0^2 - \frac{\gamma - 1}{2}\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2\right] \quad (86)$$

We need to find the solution for equation (83). satisfying boundary conditions. we obtain (83) like this form

$$\underbrace{\left(1 - \frac{U_x^2}{C^2}\right)}_{A} \frac{\partial^2 \phi}{\partial x^2} - 2 \underbrace{\frac{U_x . V_y}{C^2}}_{B} . \frac{\partial^2 \phi}{\partial x \partial y} + \underbrace{\left(1 - \frac{V_y^2}{C^2}\right)}_{C} \frac{\partial^2 \phi}{\partial y^2} = 0 \quad (87)$$

$$A \frac{\partial^2 \phi}{\partial x^2} - 2B \frac{\partial^2 \phi}{\partial x \partial y} + C \frac{\partial^2 \phi}{\partial y^2} \quad (88)$$

The terms A, B and C are functions of both the independent variables x and y and dependent variables U_x and V_y. The speed of sound C is not an additional independent variable and for our assumption of perfect gas. we obtain.

$$C^2 = \frac{\gamma - 1}{2}\left[V_{max}^2 - (U_x^2 + V_y^2)\right] \quad (89)$$

(88) relates to three types of equations, elliptic, parabolic or hyperbolic.

(33)

$$B^2 - AC = \frac{U_x^2 V_y^2}{C^4} - \left(1 - \frac{U_x^2}{C^2}\right)\left(1 - \frac{V_y^2}{C^2}\right)$$

$$= \frac{U_x^2 + V_y^2}{C^2} - 1 = \frac{V^2}{C^2} - 1$$

$$\left[B^2 - AC = M^2 - 1\right] \quad ,(90)$$

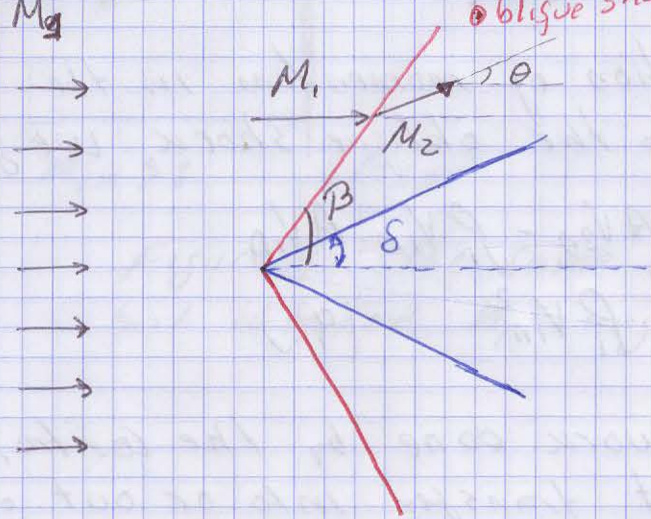
For subsonic flow $M < 1$ elliptic

For sonic flow $M = 1$ Parrobolic

For soporsonic flow $M > 1$ hyperbolic

Oblique Shocks:

Not all shock waves are normal shocks, when a space shuttle travels at supersonic speeds through the atmosphere, it produces a complicated shock pattern consisting of inclined shock waves called Oblique Shocks.



M_2 can be subsonic sonic or supersonic will depend on the upstream Mach number and turning angle.

Figure 7. An oblique shock of shock wave β formed by a slender, two-dimensional wedge of half angle δ. The flow is deflected by θ downstream of the shock and the mach number decreases.

To conserve mass, β must obviously be greater than δ, since Reynolds number of supersonic flows is typically large, the boundary layer growing along the wedge is very thin and in this analysis its effects are ignored.

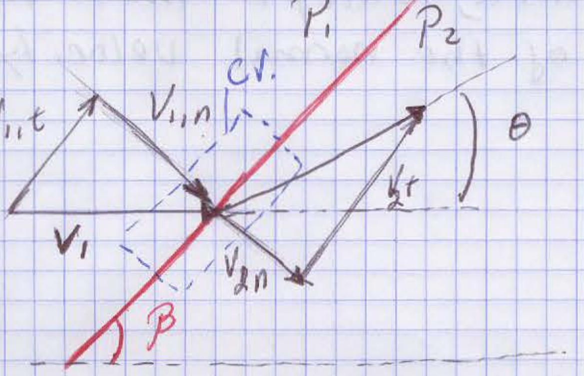


Figure 8 Velocity vector through an oblique shock of shock angle β and deflection angle θ

Conservation of mass in the control volume is given by

$$\rho_1 V_{1n} A = \rho_2 V_{2n} A \qquad (91)$$

where A is the area of the control surface that is parallel to the shock. since A is identical is dropped out.

Applying conservation of momentum in the direction normal to the oblique shock, we get

$$P_1 A - P_2 A = \rho_2 V_{2n} \cdot A V_{2n} - \rho_1 V_{1n} \cdot A V_{1n}$$

$$P_1 - P_2 = \rho_2 V_{2n}^2 - \rho_1 V_{1n}^2 \qquad (92)$$

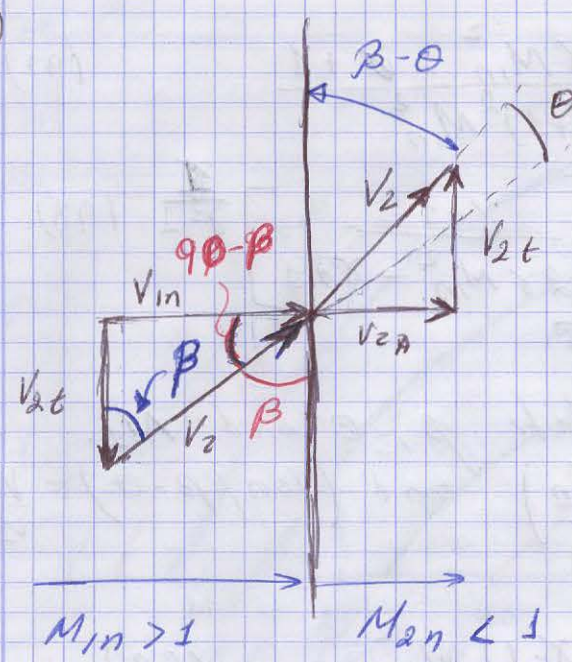
Since there is no work done by the control volume and no heat transfer into or out of the control volume, stagnation enthalpy does not change across an oblique shock and the conservation of energy

$$h_{o_1} = h_{o_2} = h_o \Rightarrow h_1 + \frac{1}{2} V_{1n}^2 + \frac{1}{2} V_{1t} = h_2 + \frac{1}{2} V_{2n} + \frac{1}{2} V_{1t}$$

we know that $V_{1t} = V_{2t}$

$$h_1 + \frac{1}{2} V_{1n}^2 = h_2 + \frac{1}{2} V_{2n}^2 \qquad (93)$$

Comparing with normal shock waves analysis; oblique shocks are identical to those across a normal shock, except that they are written in terms of the normal velocity component only.



$M_{1n} = M_1 \cdot \sin\beta$

$M_{2n} = M_2 \sin(\beta - \theta)$

where $M_{1n} = \frac{V_{1n}}{C_1}$ $M_{2n} = \frac{V_{2n}}{C_2}$

Figure 9 the same velocity vector of figure 9 but rotated by angle $\frac{\pi}{2} - \beta$, so that the oblique shock is vertical. Normal Mach numbers M_{1n} and M_{2n} are also defined.

Note: All equations, shock tables, etc, for normal shocks apply to oblique shocks as well provided that we use only the normal components of the Mach Number.

Summary of the equations

$h_{01} = h_{02}$ $T_{01} = T_{02}$

$$M_{2,n} = \sqrt{\frac{(\gamma-1) \cdot M_{1n}^2 + 2}{2\kappa M_{1n}^2 - \gamma + 1}} \quad (94)$$

$$\frac{P_2}{P_1} = \frac{2\gamma M_{1n}^2 - \gamma + 1}{\gamma + 1} \quad (95)$$

$$\frac{\rho_2}{\rho_1} = \frac{V_{1n}}{V_{2n}} = \frac{(\gamma+1) \cdot M_{1n}^2}{2 + (\gamma-1) M_{1n}^2} \quad (96)$$

$$\frac{T_2}{T_1} = \left[2 + (\gamma - 1) M_{1n}^2\right] \frac{2\gamma M_{1n}^2 - \gamma + 1}{(\gamma+1)^2 M_{1n}^2} \quad (97)$$

$$\frac{P_{02}}{P_{01}} = \left[\frac{(\gamma+1) M_{1n}^2}{2 + (\gamma-1) M_{1n}^2}\right]^{\frac{\gamma}{\gamma-1}} \left[\frac{\gamma+1}{2\gamma M_{1n}^2 - \gamma + 1}\right]^{\frac{1}{\gamma-1}} \quad (98)$$

Also we have to relate β, θ and M_1.
We know that $\left(\tan\beta = \frac{V_{1n}}{V_1}\right)$ and $\left(\tan(\beta-\theta) = \frac{V_{2n}}{V_{2t}}\right)$

$$\frac{V_{2n}}{V_{1n}} = \frac{\tan(\beta-\theta)}{\tan\beta} = \frac{2 + (\gamma-1).M_{1n}^2}{(\gamma+1) M_{1n}^2} \quad (99)$$

$$\frac{V_{2n}}{V_{1n}} = \frac{2 + (\gamma-1) M_1^2 \sin^2\beta}{(\gamma+1) . M_1^2 \sin^2\beta} \quad (100)$$

Applying trigonometric indentities for $\cos 2\beta$ and $\tan(\beta-\theta)$

$$\cos(2\beta) = \cos^2\beta - \sin^2\beta$$

$$\tan(\beta-\theta) = \frac{\tan\beta - \tan\theta}{1 + \tan\beta . \tan\theta}$$ we obtain.

$$\tan\theta = \frac{2\cot.\beta\,(M_1^2 \sin^2\beta - 1)}{M_1^2 (\gamma + \cos 2\beta) + 2} \quad (101)$$

the deflection angle θ is function of the shock angle β and specific heat ratio and Mach number M_1 (upstream).

Next figure displays the full range of possible shock waves at a given free-stream Mach number.

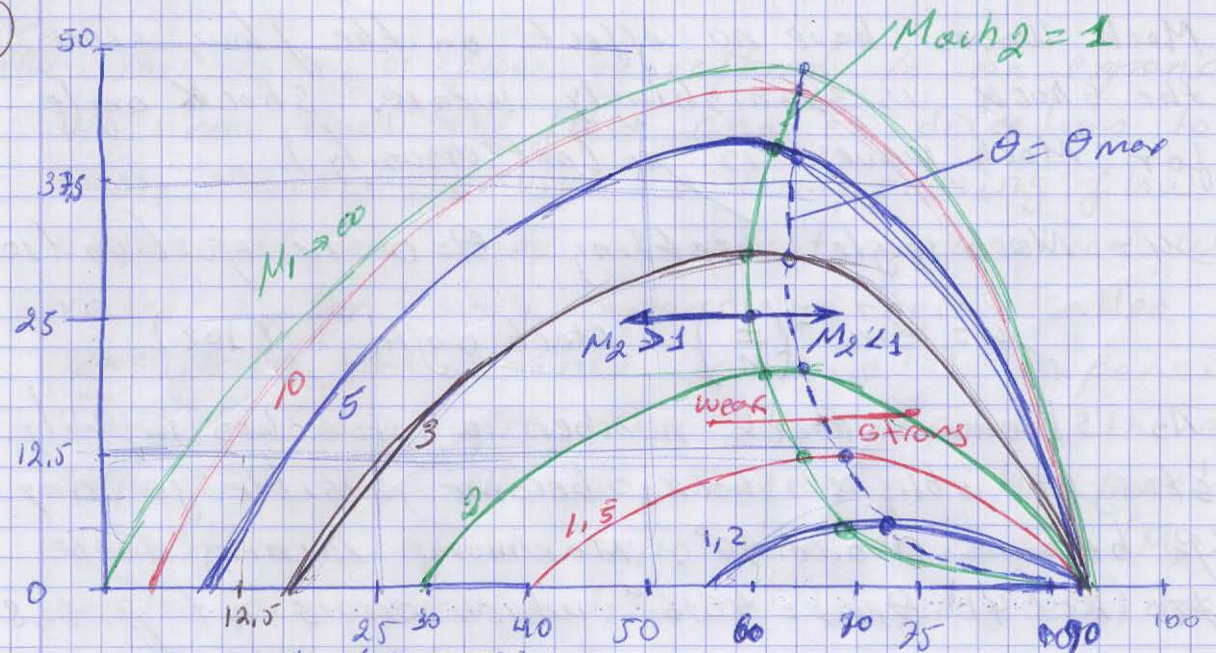


Figure 10 (θ vs β)
β between $0°$ and $90°$

There are two possible values of β for $\theta < \theta_{max}$ the dashed blue lino passes through the locus of θ_{max} values, dividing the shock into weak oblique shocks (smaller values of β) and strong oblique shocks (Higher values of β), At given value of θ, the weak shocks are more common and is preferred by the flow unless the dowstream pressure conditions are high enough for the formation of a strong shock.

For a given upstream Mach number, M_1, there is a unique value of θ for which the dowstream Mach number M_2 is equal to 1. The green line passes through the locus of values where $M_2 = 1$, to left the Mach ($M_2 > 1$) and to the right line $M_2 < 1$

For a given upstream Mach number there are two shock angles where there is no turning of the flow. The strong case $\beta = 90°$ corresponds to a normal shock, and the weak case $\beta = \beta_{min.}$ represents the weakest possible oblique shock at that Mach number (MACH WAVE).

Mach waves have no effect on the flow, since the shock is vanishingly weak. Shock angle for Mach waves is μ (not viscosity).

μ = Mach angle, setting $\theta = 0$ in equation (101)

$$\mu = \sin^{-1}\left(\frac{1}{M_1}\right) \quad \text{Mach angle.} \quad (102)$$

- As stream Mach number approaches infinity straight oblique shocks become possible for any β between 0 and 90°. Maximum turning angle for ($K=1.4$) $\theta_{max} = 45,6°$ which occurs at $\beta = 67,8°$

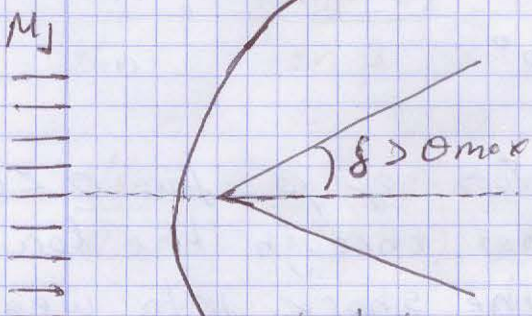


Figure 11
Detached oblique shock or bow wave

A detached oblique shock occurs upstream of a two-dimensional wedge of half angle δ when δ is greater than the maximum possible deflection angle θ. A shock of this kind is called a bow wave because of its resemblance to the water that forms at the bow of a ship

2.5.1 Prandtl - Meyer Expansions Waves

We are going to study situations where supersonic flow is turned in the opposite direction. (See figure 12.)

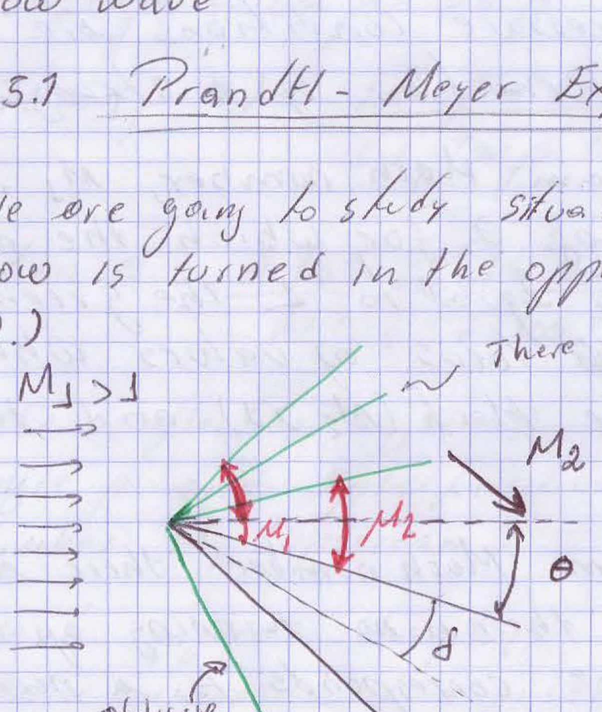


Figure 12 Expansion waves over a wedge.

(10). This type of flow is reffered to an expanding flow. We know that the flow changes direction to conserve mass, However unlike a comppessing flow, an expanding flow does not result in a shock waves, rather a continuos expanding region called "expansion fan". Since individual expansion wave is isentropic, the flow across the entire expansion fan is also isentropic. The Mach number downstream of the expansion increases ($M_2 > M_1$) while presure, density and temperature decrease,

Prandtl-Meyer expansion waves are inclined at local Mach angle μ.

$$\text{first Mach angle } \mu_1 = \sin^{-1}\left(\frac{1}{M_1}\right) \qquad (103)$$

$$\mu_2 = \sin^{-1}\left(\frac{1}{M_2}\right)$$

If we neglecte the influence of the boundary layer along the wall. How we calcolate M_2? we need to calculate first the deflection angle θ.

$$(104) \quad \theta = \nu(M_2) - \nu(M_1)$$

ν { angle called Prandtl-Meyer function.

$$(105) \quad \nu(M) = \sqrt{\frac{\gamma+1}{\gamma-1}} \, \tan^{-1}\left(\sqrt{\frac{\gamma-1}{\gamma+1}(M-1)}\right) - \tan^{-1}\left(\sqrt{M-1}\right)$$

Physically ν is an angle through which the flow must expand. starting with $\nu = 0$ at $Ma = 1$

To calcolate M_2, first we calcolate $\nu(M_1)$ from (105) and $\nu(M_2)$ with (104) and finally M_2 with (105). this is true because we consider isentropic flow. on expansion fan.

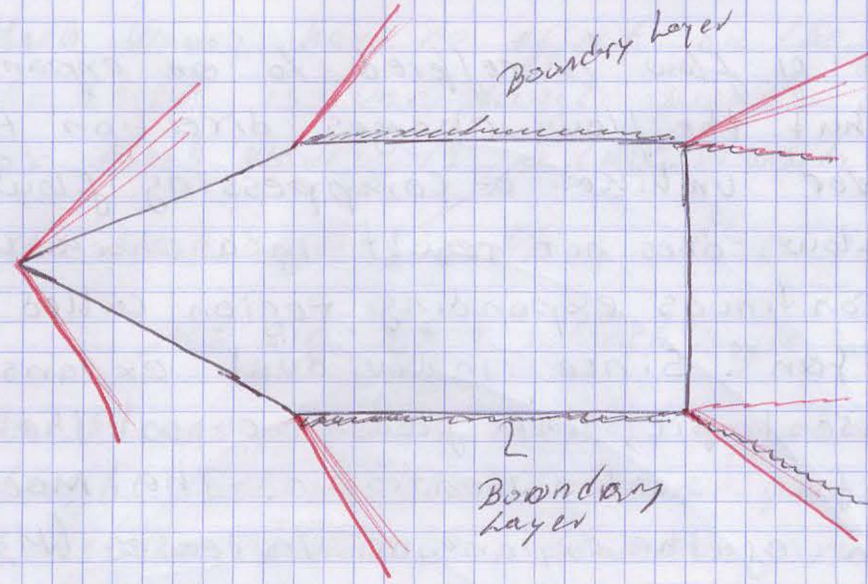


Figure 13. A cone-cylinder of 12.5° half-angle in a Mach number 1.84 flow. The boundary layer becomes turbulent shortly downstream of the nose. Expansion waves are seen at the corners and at the trailing edge of the cone.